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The transformation of a solitary wave over an uneven bottom

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Based on a set of approximate equations for long waves over an uneven bottom, numerical results show that as a solitary wave climbs a slope the rate of amplitude increase depends on the initial amplitude as well as on the slope. Results are also obtained for a solitary wave progressing over a slope onto a shelf. On the shelf a disintegration of the initial wave into a train of solitary waves of decreasing amplitude is found. Experimental evidence is also presented.

1. Introduction

In recent studies of long waves in shallow water, the interplay between non-linearity and dispersiveness has received much attention (Whitham 1967). Added impetus to the current interest in this topic is due to the occurrence of analogous situations in widely different physical phenomena (see e.g. Gardner *et al.* 1967). Aside from classical theories for permanent waves (cnoidal and solitary) and an analytical theory announced by Gardner *et al.* (1967) pertaining to the independence of solitary waves, initial value problems have been studied numerically by Long (1964) and Peregrine (1966) in the context of water waves and by Zabusky and co-workers in plasma physics (see e.g. Zabusky & Kruskal 1965; Zabusky 1967). The governing equation in these works is the Korteweg–de Vries equation, or a simple extension thereof, thus corresponding to waters of constant depth. The effect of an uneven bottom on water waves of this class is of obvious engineering importance, and has been incorporated in the governing equations by Mei & LeMéhauté (1966) and by Peregrine (1967). Only Peregrine (1967) obtained quantitative results using a finite difference scheme to compute the deformation of a solitary wave climbing a beach. The same problem is first treated here with a more appropriate initial condition than Peregrine's; additional results are obtained which are in better corroboration with the experiments of Ippen & Kulin (1955), Kishi & Saeki (1966) and Camfield & Street (1969).

The present paper also treats the related problem of a solitary wave propagating from a channel of constant depth, past a mild slope, onto a shelf of constant but smaller depth. When progressing over the shelf the wave is seen to disintegrate into a train of solitary waves of decreasing amplitudes, which is in qualitative agreement with experiments.

2. The governing equations

Based on potential theory the appropriate equations have been derived by Mei & LeMéhauté (1966) and in a different but equivalent version also by Peregrine (1967).

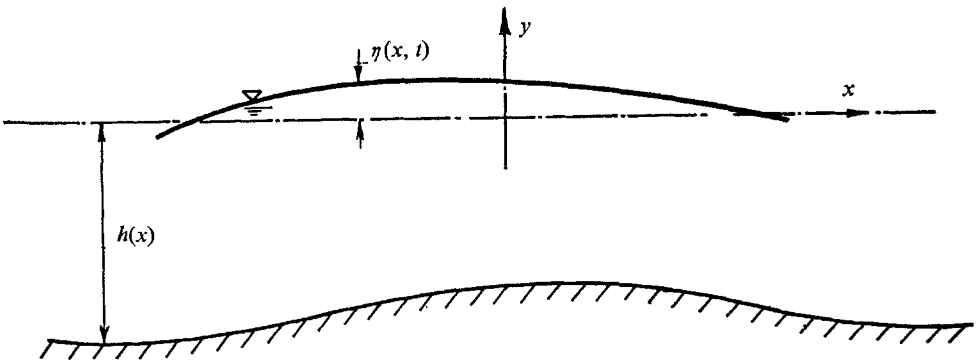

FIGURE 1. Definition of symbols.

We adopt dimensionless variables, as defined below and in figure 1,

$$\left.\begin{aligned}(\eta^*, x^*, y^*) &= L(\eta, x, y),\\ t^* &= (L/\sqrt{(gh_0)})\,t,\\ p^* &= (\rho g h_0)\,p,\end{aligned}\right\} \tag{2.1}$$

where variables on the left-hand sides are dimensional and those on the right-hand sides dimensionless. h_0 and L are typical vertical and horizontal length scales respectively.

For shallow water and slow variation of the bottom profile, i.e.

$$h, h', h'', \text{etc.} = O(\epsilon), \quad \text{where} \quad \epsilon = h_0/L \ll 1 \tag{2.2}$$

and for waves of the cnoidal class, i.e. (Ursell 1953),

$$\eta/\epsilon^3 = O(1), \tag{2.3}$$

the set of approximate equations obtained by Mei & LeMéhauté are

$$\frac{\partial \eta}{\partial t} + \frac{\partial[u(h+\eta)]}{\partial x} - \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} = Au + B\frac{\partial u}{\partial x} + \tfrac{3}{2}h^2h'\frac{\partial^2 u}{\partial x^2} + O(\epsilon^6) \tag{2.4}$$

and
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\epsilon}\frac{\partial \eta}{\partial x} - \tfrac{1}{2}h^2\frac{\partial^3 u}{\partial t\,\partial x^2} = [(h')^2 + hh'']\frac{\partial u}{\partial t} + 2hh'\frac{\partial^2 u}{\partial t\,\partial x} + O(\epsilon^5), \tag{2.5}$$

where
$$\left.\begin{aligned}u &= \text{horizontal velocity at the bottom},\\ A &= (h')^3 + 3hh'h'' + \tfrac{1}{2}h^2h''',\\ B &= 3h(h')^2 + \tfrac{3}{2}h^2h''.\end{aligned}\right\} \tag{2.6}$$

For horizontal bottom or if h', h'' etc. $= o(\epsilon)$ the right-hand sides can be ignored and the equations reduce to those derived by Lin & Clark (1959) and by Long (1964). The terms involving the highest derivatives are not unique but may be rewritten using the first-order relationships, as was first done by Long (1964), to give fixed characteristics, which facilitate numerical computations.

Correcting some minor algebraic mistakes made by Mei & LeMéhauté (1966) these equations are in characteristic form

$$\frac{d\eta}{dt} = v \quad \text{and} \quad \frac{dv}{dt} = a, \tag{2.7}$$

along the coinciding characteristics $x =$ constant, and along the two distinct characteristics,

$$\frac{dx}{dt} = \pm C_0(x) = \pm \sqrt{\left(\frac{3h}{\epsilon}\frac{1-(h')^2-hh''}{1-\frac{1}{2}hh''}\right)}, \tag{2.8}$$

$$\left(\frac{ch}{6}+\frac{uh}{2}\right)\frac{d\eta}{d\beta}+\tfrac{5}{12}h^2h'\frac{dv}{d\beta}+(1-\tfrac{1}{2}hh'')\frac{\epsilon hc^2}{6}\frac{du}{d\beta}+\frac{\epsilon ch^2}{12}\frac{da}{d\beta}$$
$$= \left(\frac{ch}{2}D+\frac{\epsilon h'c^2}{6}u\right)u+\frac{hh'}{6}(\tfrac{5}{2}h-\epsilon c^2)\,a+\left(\frac{\eta c}{2}-\frac{hc}{3}+\left(\frac{h}{2}+\frac{\epsilon c^2}{6}\right)u\right)v, \tag{2.9}$$

where in (2.9)
$$D = \frac{h'}{h}\eta - h' + (h')^3 + \tfrac{1}{3}h^2h''' + 2hh'h'' \tag{2.10}$$

and
$$c = \begin{pmatrix} C_0(x) \\ -C_0(x) \end{pmatrix} \quad \text{along} \quad \beta = \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \frac{1}{2}\left(t \mp \int \frac{dx}{C_0(x)}\right) = \text{constant}. \tag{2.11}$$

These equations are solved numerically, as described in the appendix, using the IBM 360 computer.

3. The transformation on a slope

Instead of a closed beach at the end of a horizontal channel, it is necessary in our formulation to adopt the bottom depicted in figure 2 with a small but finite h_1, say $h_1 = 0{\cdot}1h_0$. Since no significant disturbance had reached the upper end of the slope, when the computation was stopped at a place limited by numerical stability (appendix), all results reported here are applicable to a closed beach.

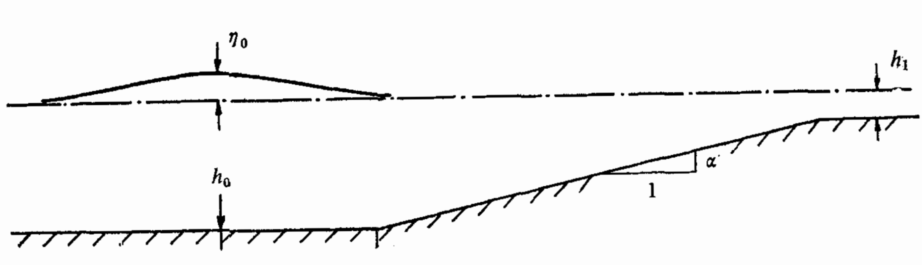

FIGURE 2. Geometry of the problem.

At $t = 0$, the crest of a solitary wave of amplitude η_0 is supposed to pass $x = 0$, far from the slope, and initial values of η, u, v and a are taken from the classical formulas appropriate for the constant depth h_0. The distance between the point $x = 0$ and the bottom of the slope x_B is sufficiently large as long as

$$\eta(x_B, 0)/\eta_0 < 0{\cdot}1.$$

Peregrine (1967) in a similar study adopted a different initial condition such that at $t = 0$, a solitary wave starts with its crest immediately above the start of the slope corresponding to $x_B = 0$. This initial condition seems rather artificial and causes some difference from our results.

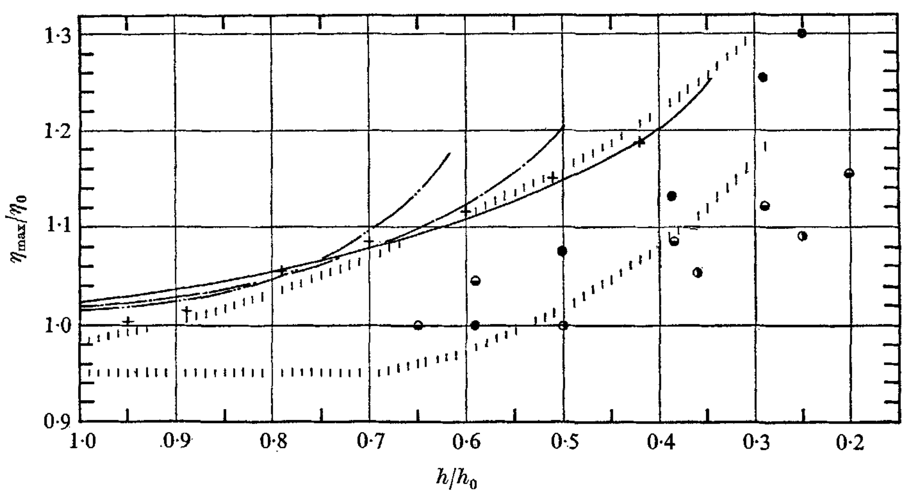

FIGURE 3. Amplitude variation with depth for beach slope $= \frac{1}{20}$. Calculated results for η_0/h_0: ——, 0·1; —·—, 0·15; ---—, 0·2. +, Peregrine's result for $\eta_0/h_0 = 0{\cdot}1$. Experiments for $\eta_0/h_0 = 0{\cdot}25$–$0{\cdot}68$ by Ippen & Kulin between shaded lines. Experiments by Kishi & Saeki for η_0/h_0: ◑, 0·043; ⊖, 0·12; ●, 0·305.

For the slope, $\alpha = 1:20$, the transformation of solitary waves of different initial amplitudes were studied. Figure 3 shows the variation of maximum amplitude, η_{max}, with depth. The largest values of η_{max}/h are all within the confidence limits given in table 3 (appendix).

In addition to the usual steepening of the front, our results indicate in general that when a wave reaches the slope the amplitude has increased slightly, and as it climbs the slope the amplitude increases at a rate increasing with the value of η_{max}/h, thus giving a transformation depending upon the initial amplitude.

The reflexion from the beach is accurately predicted by the analytical approximation of Peregrine (1967), i.e. it is of the form of a nearly constant elevation of the height $\frac{1}{2}\alpha(\frac{1}{3}\eta_0/h_0)^{\frac{1}{2}} h_0$.

In their experiments with the beach of slope 1/20, Ippen & Kulin (1955) found consistently that the amplitude decreases from its initial value measured some distance away from the slope, to a smaller value as the crest reaches the slope. Our calculations show the opposite trend. In fact according to inviscid theory a slight increase in amplitude should be expected by the following argument. Clearly as the first 'half' of the solitary wave advances onto the slope the front of the reflexion is created. Associated with this positive elevation are particle velocities in the direction of propagation, i.e. against the advancing solitary wave. Thus the main crest will act as if superposed on an opposing current, which calls for a slight increase in amplitude in order to maintain roughly the same transport of energy. The observed decrease can be attributed to frictional attenuation, when the latter is estimated according to the empirical findings of Ippen, Kulin & Raza (1955). Peregrine's initial condition precludes any conclusion either way in this regard.

Neither Ippen & Kulin nor Peregrine reported a systematic change in amplitude variation with the initial amplitude. However, experiments by Kishi & Saeki (1966) with a roughened bottom (slope 1:20) show the similar trends as our results. Because of the rough bottom the frictional effects are pronounced, and the amplitude does not increase appreciably over the first part of the slope. However, as the amplitude to depth ratio becomes large the amplitude increase is no longer cancelled by the frictional attenuation. Thus, even with the scatter shown in figure 3, these experiments show distinctively a dependence upon initial amplitude, in qualitative corroboration with our inviscid theory.

No dependence of amplitude variation on bottom slope was reported by Peregrine (1967), However, Ippen & Kulin found the following empirical relationship (although with considerable scatter),

$$\eta_{\max}/\eta_0 = K(h/h_0)^{-n}, \tag{3.1}$$

where K is a constant and n depends on the bottom slope as shown in table 1. Comparison with the present numerical results ($\eta_0/h_0 = 0{\cdot}1$) given in the same table and figure 4 indicates the same trend. However, since the experiments correspond to values of $\eta_0/h_0 = 0{\cdot}25$–$0{\cdot}68$ the agreement can only be considered qualitative.

Slope	0·065	0·05	0·023
Ippen & Kulin (1955)	0·19	0·26	0·47
Present computation	0·15	0·18	0·30

TABLE 1. Comparison of n values in equation (3.1)

Recent experiments by Camfield & Street (1969) reported amplitude variation of a solitary wave climbing a smooth beach. In figure 4 their data for a small slope 0·02 is seen to agree well with our theory for the slope 0·023. Their results for a different slope (0·045), when plotted in terms of variables adopted here, show a scatter even larger than that of Ippen & Kulin (1955). For this reason, a comparison with computed results for this slope is omitted.

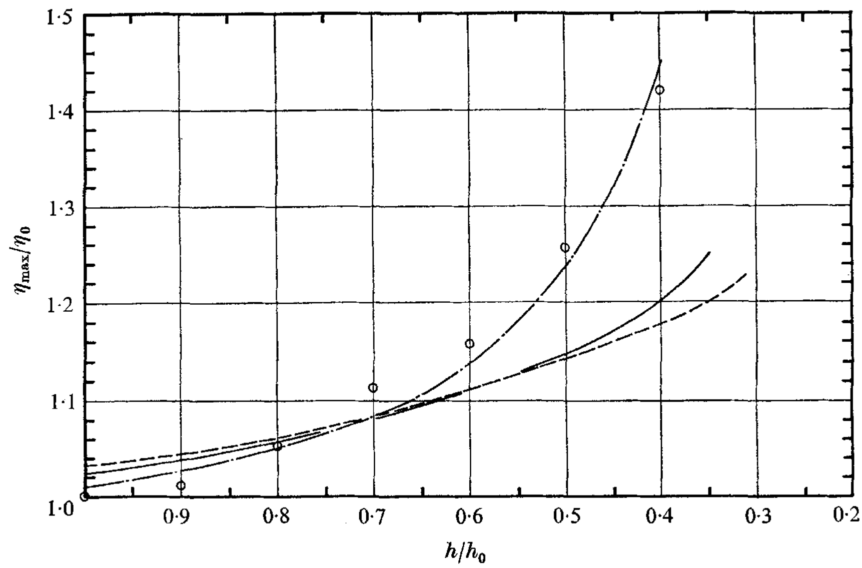

FIGURE 4. Amplitude variation with depth for initial amplitude to depth ratio, $\eta_0/h_0 = 0 \cdot 1$. Calculated results for slopes, α: ——–, 0·065; ———, 0·05; —·—, 0·023. ○, experiments by Camfield & Street for $\alpha = 0 \cdot 02$.

4. The scattering of a solitary wave by a shelf

The depth on the shelf, h_1, is now changed to $0 \cdot 5\, h_0$ but otherwise the same geometry is kept as shown in figure 2.

The profiles for an initial amplitude to depth ratio of 0·12 and a slope $\alpha = 1:20$, are shown in figure 5. As could be expected the development of the wave up to the point where it enters the shelf is the same as found in § 3, except for the last part of the slope. When the wave, distorted by climbing the slope, enters the shelf, it undergoes a rather unexpected sequence of transformations. Figures 5 (*a*) and (*b*) show the profile becoming more peaked as the maximum amplitude increases at a rate decreasing with distance travelled on the shelf. Then a hump of smaller height appears at the rear and gradually trails behind the main wave. While the main wave outruns the second hump, its tendency of growth is eventually arrested. The latter then experiences a development similar to that of the main wave, giving birth to a third hump. This third hump, however, is followed by a train of oscillatory waves, having the same characteristics as the linear shallow water waves. A tendency for the two larger peaks to separate into two solitary waves can be noticed. This is supported by the comparison in figure 5 (*c*) between the computed profiles and the profiles of theoretical solitary waves of equal amplitudes. A corresponding comparison of wave speeds is shown in table 2, with a discrepancy consistent with that noted in table 3 (appendix).

The reflected wave is again precisely of the form predicted by Peregrine's theory, with a length approximately twice that of the slope.

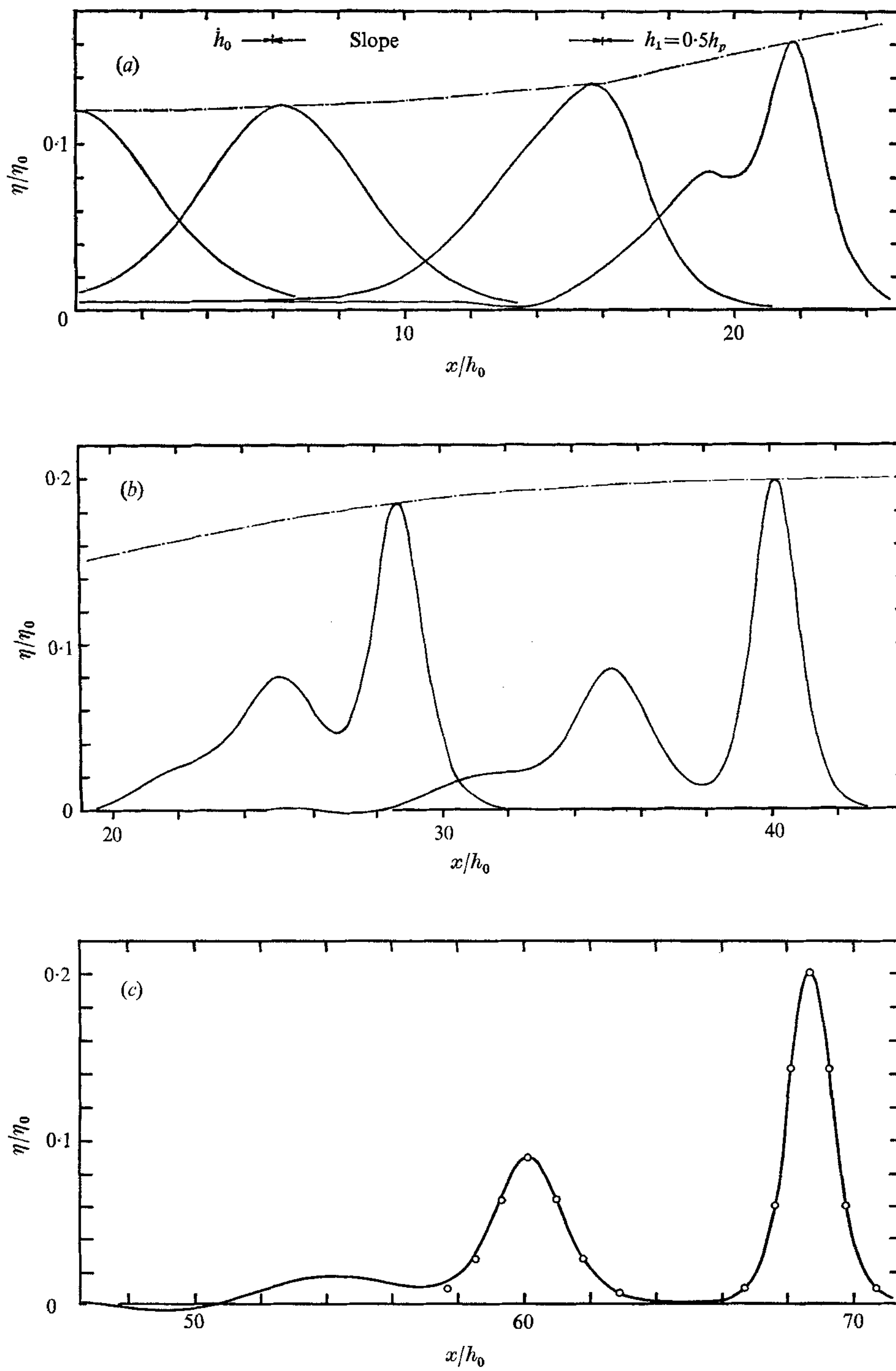

FIGURE 5. Transformation of a solitary wave ($\eta_0/h_0 = 0{\cdot}12$) propagating over a slope, $\alpha = \frac{1}{20}$, onto a shelf of smaller depth, $h_1 = 0{\cdot}5\,h_0$. —·—, indicates the growth of the main crest. In (*c*), ○, indicates theoretical profiles of solitary waves.

	First crest	Second crest
η_{max}/h_1	0·4	0·18
Solitary wave theory	0·838	0·768
Present computation	0·891	0·778

TABLE 2. Comparison of wavespeeds

5. Experimental evidence of solitary wave disintegration

Street, Burges & Whitford (1968) were the first to report the formation of undulations behind a solitary wave on the upper part of a shelf. For different depth ratios, h_1/h_0, their results indicate that the formation of the secondary peaks is more rapid the larger the amplitude to depth ratio on the shelf. In the case corresponding to our computations, $h_1/h_0 = 0{\cdot}5$, their experiments are all performed with amplitudes too small to give the formation of secondary peaks within the length of their tank; therefore no comparison of this feature can be made. However, a quantitative comparison of the maximum amplitude, $\eta_{\max}$, observed at a station $12h_0$ onto the shelf ($x = 28h_0$ in figure 5(*b*)), can be made by extrapolating their findings to correspond to $\eta_0/h_0 = 0{\cdot}12$. Their result $\eta_{\max}/n_0 \approx 1{\cdot}58$ compares favourably with our computed value 1·52.

Since the experiments by Street *et al.* (1968) do not furnish detailed information for the disintegration on the shelf, our own results from some simple tests are presented. With the existing lucite flume it was necessary to reduce the water depth to 3 in. (1·5 in. on the shelf) in order to make the flume effectively long. A solitary wave was created by releasing a plunger vertically, and when the main wave has passed an absorber, the trailing disturbances were cut off by dropping a plate vertically. Four resistance probes were installed along the channel, as shown in the sketch in figure 6.

In spite of the small depth, and the rather crude way of generating the solitary wave, cases where $\eta_0/h_0 > 0{\cdot}1$ show agreement with the theoretical profiles, at the first probe as shown in figure 6(*a*) for $\eta_0/h_0 = 0{\cdot}12$. The neglect of viscous damping in the theory prohibits quantitative agreement at the later stations. The appearance of the second crest, is nevertheless, evident.

To estimate the influence of friction, we make use of the experimental results of Ippen *et al.* (1955), for the damping of solitary waves on water of constant depth. Their empirical formula for the amplitude attentuation was given as follows:

$$\left[\left(\frac{\eta_{(x)}}{h}\right)^{-\frac{1}{4}} - \left(\frac{\eta_0}{h}\right)^{-\frac{1}{4}}\right] = K\frac{x}{h}, \tag{5.1}$$

$$x = \text{distance travelled}, \quad \eta_0 = \text{amplitude at } x = 0,$$

$$\eta_{(x)} = \text{amplitude at } x = x, \quad K = \text{damping coefficient}. \tag{5.2}$$

To achieve an order estimate, we assume that the empirical damping coefficient so found can be used for the attenuation of the first crest of our distorted wave. Thus we find the estimated amplitude which would have been recorded in the experiments had there been no friction (indicated by $\otimes$ in figure 6). Though not

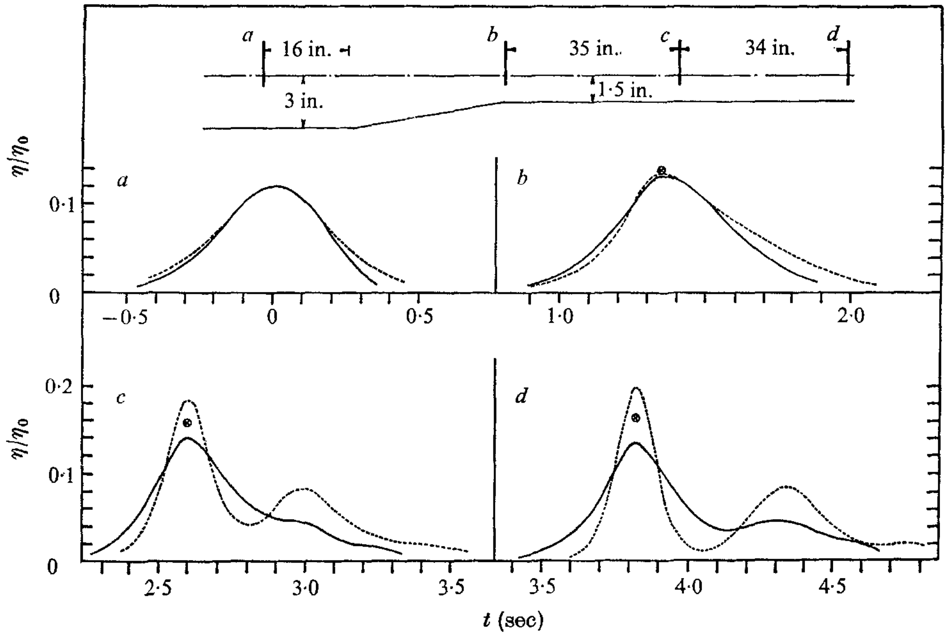

FIGURE 6. Comparison between: ----, theory and ——, experiment for initial amplitude to depth ratio, $\eta_0/h_0 = 0{\cdot}12$. ⊗, estimated experimental amplitude for no viscous damping.

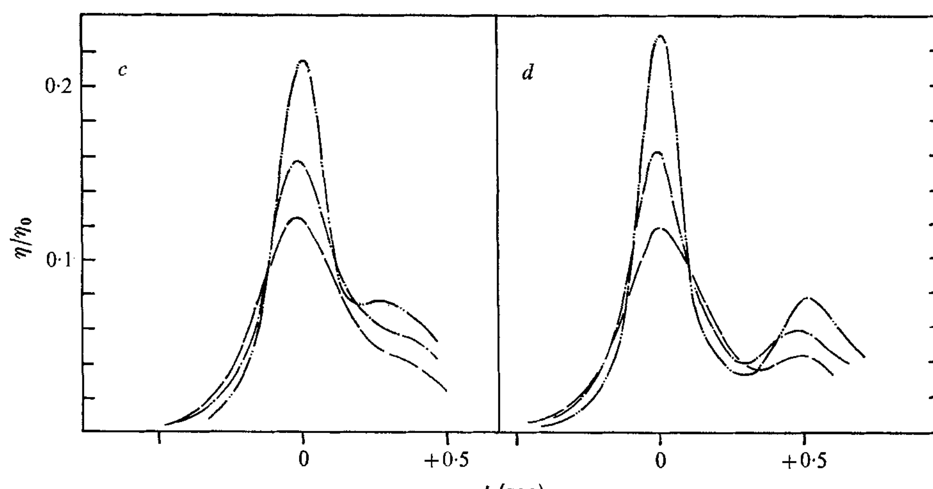

FIGURE 7. Observed development at stations c and d for different initial amplitude to depth ratios. η_0/h_0: — —, 0·11; —·—, 0·14; —··—, 0·17.

accounting for the entire discrepancy, this estimate does show that viscous damping is the main source of error. The later appearance of the second hump can be explained by the amplitude in the experiment being smaller than in the theory.

According to Ippen *et al.* (1955), the damping of a solitary wave decreases with increasing amplitude. In figure 7 we show the development, which took place at probes c and d for waves of different initial amplitudes. Clearly the agreement with theory is better for larger initial amplitudes (or smaller viscous influence). In none of the experiments was the tank long enough to observe the final stages of disintegration from undulations to a train of solitary waves.

6. Concluding remarks

Based on the Korteweg–de Vries equation and spatially sinusoidal initial data, Zabusky & Kruskal (1965) found numerically that a steepening of each crest was followed by a disintegration into a series of solitary waves, (called 'solitions' by them), which interact with those from the neighbouring periods in a complicated manner. Their results have been applied by Zabusky & Galvin (1968) in an attempt to predict the secondary crests observed in a wave channel of constant depth (Horikawa & Wiegel 1959, Galvin 1968). For the variable depth Byrne (1969) recently reported some field evidence of secondary crests, when periodic waves of the cnoidal class pass over a submerged sand bar. All these cases and the results of §§4 and 5 share a common feature, i.e. prior to its disintegration a wave crest is steeper at the front and flatter at the back. As the stepped bore may be regarded as a limiting configuration of this kind (with a horizontal back), Peregrine's (1966) physical interpretation of the development of an undular bore from a stepped bore should be pertinent to all cases mentioned above.

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Appendix. Computational aspects

The small parameter, $\epsilon = h_0/L$, pertains to the fluid motion, and involves the choice of a horizontal length scale. If we adopt the distance between the two points of a solitary wave where $\eta/\eta_0 = 0{\cdot}001$ to be the 'wavelength' L, the classical formula then gives

$$L/h_0 \cong 10\sqrt{(h_0/\eta_0)}. \tag{A 1}$$

For $0{\cdot}1 < \eta_0/h_0 < 0{\cdot}4$ we have $\frac{1}{30} < \epsilon < \frac{1}{15}$, and ϵ is seen to be of the same order of magnitude as the slopes, α, studied herein. We have therefore chosen $\epsilon = \alpha$.

In the computations initial data are prescribed for $t = 0$ corresponding to the classical formulas for a solitary wave. Characteristic grids are arranged in the x, t plane so that $\Delta t =$ constant; as shown by (2.8) this requires a variable Δx on the slope. Note that $\Delta x = \sqrt{(3h/\epsilon)}\,\Delta t$ when h is constant. Central differences are used to replace derivatives along the characteristics. The variable coefficients are replaced by the mean of the values at two neighbouring grid points along the pertinent characteristic. The equations are solved iteratively, with two iterations always sufficient.

Numerical accuracy is checked for three aspects: (i) Total mass conservation (at all time within 1 %). (ii) Stability with the criterion that on constant depth a solitary wave propagates for $100\Delta t$ without significant change in shape. Table 3 shows the dependence of maximum usable stepsize upon the wave amplitude, the ratio of computed wave speed, c_c, and the theoretical $c_t = \sqrt{(1+\eta_0/h_0)}$ and the nature of instability. Accordingly, computations are carried out only as long as the local amplitude to depth ratio is less than 0·4. It may be remarked that during the time $100\Delta t$ the wave travels a distance slightly larger than the characteristic length as defined by equation (A 1). (iii) The grid size in the region $h = h_0$ is $\Delta x = 0{\cdot}15h_0$ ($\Delta x = \sqrt{3}\Delta t$) in all cases presented and sample checks for convergence were made by changing it to $\Delta x = 0{\cdot}1\,h_0$.

η_0/h_0	$(\Delta x/h_0)_{\max}$	c_c/c_t	If $\dfrac{\Delta x}{h_0} > \left(\dfrac{\Delta x}{h_0}\right)_{\max}$
0·1	0·6	1·00	η_0 decreases
0·2	0·5	1·01	
0·3	0·4	1·03	
0·4	0·25	1·06	
0·5	Too small	—	η_0 increases

TABLE 3. Stability restriction on grid size for various amplitude to depth ratios

In deriving the basic approximate equations, existence of higher derivative h'' and h''' are required. This cannot in principle be achieved by the broken line geometry shown in figure 2. However, using a sinusoidal transition with the same average slope as that of the plane slope gives only insignificant changes in the transmitted wave, whereas the reflected wave is basically different and depends strongly on the local value of the slope, as may be inferred from Peregrine (1967).

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